J. Fluid Mech. (1981), *vol.* 110, *pp.* 149–159
Printed in Great Britain

The stability of a periodically heated layer of fluid

By JOHN DOWDEN

Department of Mathematics, University of Essex, Colchester C04 3SQ, U.K.

(Received 2 October 1979 and in revised form 20 December 1980)

The stability of an infinite layer of fluid of uniform thickness at rest with two horizontal boundaries is investigated when the difference between the temperatures at the top and bottom boundaries has a component which is a fluctuating periodic function of time in addition to a constant part. When both boundaries are free small fluctuations have a stabilizing effect on the layer, while large fluctuations tend to make it less stable, consistently with the numerical results of Yih & Li (1972); the effect is attributable entirely to the variation of the temperature gradient with time. An approximate relation between the mean Rayleigh number and the amplitude of the fluctuations is found which separates stable situations from unstable ones. This is compared with the criteria deduced by Homsy (1974) using energy arguments for disturbances of any amplitude.

1. Introduction

When a layer of fluid is heated from below or cooled from above in a manner which gives a periodic change in the temperature gradient in time and consequent variations in the temperature distribution vertically, the conditions under which it becomes unstable are not necessarily the same as in the classical Bénard experiment with a constant temperature gradient (Graham 1933; Chandra 1938; Sutton 1950). This and related problems have received a substantial amount of attention, and the subject of modification of stability limits in modulated flows has been reviewed by Davis (1976). In the case of the Bénard problem, when the applied frequency is very low the effect of a small-amplitude modulation is to stabilize the layer, but what happens as the amplitude is increased is less certain. The numerical work of Yih & Li (1972) shows that the layer reaches a maximum degree of stability as the modulation is increased, after which it becomes less stable. Their calculations were carried out for moderate values of the applied frequency so that there was substantial variation of the temperature gradient in space as well as in time. Since Currie (1967) has shown that a variation in space alone can sometimes have a stabilizing effect it is not clear whether the existence of this optimum level of modulation was a simple consequence of the periodicity of the temperature gradient, or a result of the accompanying variation in space.

Homsy (1974) used the method of energy to produce two stability criteria for a large class of Bénard problems, both applicable to disturbances of arbitrary amplitude. The first was 'strongly global stability', which he defined to mean that the energy decays monotonically and exponentially; the second is 'asymptotic stability', which means that the energy decays asymptotically to zero over many cycles of modulation. He found that in many cases the energy and linear stability theory limits lie close to one another.

It is the purpose of this paper to investigate whether the reduction of stability for large-amplitude modulation can be accounted for without having to appeal to the effect of spatial variation in the temperature gradient, and to relate the result to Homsy's stability criteria. To do this the problem of a layer of fluid with two free surfaces periodically heated at low frequency will be considered in detail; the variation in time is then so slow that the temperature gradient is almost constant in space at any given instant, but superimposed on this steady cooling is a periodic fluctuation. The problem has been studied by Homsy, and by Rosenblat & Herbert (1970) when the amplitude of modulation is small, and it will be shown here that the analysis may be extended to cover larger amplitudes as well.

2. Analysis

The equations of motion for a layer of fluid in which the mean temperature gradient is not a constant but a function of time were first obtained by Goldstein (1959). Suppose the layer of fluid has mean density ρ_0, kinematic viscosity ν, thermal diffusivity κ and coefficient of expansion α, with temperature T and components (u', v', w') of the velocity vector $\mathbf{u}'$ at a point with co-ordinates (x', y', z') at time t'. If the temperature distribution is $\overline{T}$ in a solution with no mean velocity of the full set of equations governing the motion of the layer then, following Rosenblat & Herbert, equations governing small perturbations in $\mathbf{u}'$ and T can be found. If the solutions for the vertical component of $\mathbf{u}'$ and for $T-\overline{T}$ are sought equal to the real parts of

$$W(z', t')\exp\{i(k_1x'+k_2y')\}, \quad \Theta(z', t')\exp\{i(k_1x'+k_2y')\}$$

respectively, where $W(z', t')$ and $\Theta(z', t')$ are complex functions of z' and t', these equations are

$$\frac{\partial}{\partial t'}\left(\frac{\partial^2}{\partial z'^2}-k^2\right)W = -g\alpha k^2\Theta+\nu\left(\frac{\partial^2}{\partial z'^2}-k^2\right)^2 W, \tag{1}$$

$$\frac{\partial\Theta}{\partial t'} = -W\frac{\partial\overline{T}}{\partial z'}+\kappa\left(\frac{\partial^2}{\partial z'^2}-k^2\right)\Theta, \tag{2}$$

where $k^2 = k_1^2+k_2^2$. If the lower boundary of the layer is held at a constant temperature T_0, and the surface of the layer, $z' = h$, is cooled to a prescribed but fluctuating temperature $T_0-T_1-T_2\cos\Omega' t'$, $\overline{T}$ is given by the real part of

$$T_0-T_1z'/h-T_2\frac{\sinh(z'(i\Omega'/\kappa)^{\frac{1}{2}})}{\sinh(i\Omega'/\kappa)^{\frac{1}{2}}}e^{i\Omega' t'}$$

(Rosenblat & Herbert, equation (2.13)).

In order to see the general characteristics of the dependence of the solution on time rather than space, we shall here only consider the case when

$$h\surd 2\kappa \ll \Omega'^{-\frac{1}{2}}. \tag{3}$$

T is then given approximately by $T_0-z'(T_1+T_2\cos\Omega' t')/h$, so that, provided condition (3) is approximately satisfied, the temperature gradient is given by

$$\frac{\partial\overline{T}}{\partial z'} = -(T_1+T_2\cos\Omega' t')/h.$$

With this approximation for the temperature gradient, put (1) and (2) in dimensionless form by writing

$$z' = zh, \quad t' = h^2t/\nu, \quad \Omega' = \nu\Omega/h^2, \quad k^2 = a^2/h^2, \quad R = \alpha g T_1 h^3/\kappa\nu, \quad S = \alpha g T_2 h^3/\kappa\nu.$$

Equation (1) is now

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)\left(\frac{\partial^2}{\partial z^2}-a^2-\frac{\partial}{\partial t}\right) W = \frac{\alpha g h^2 a^2}{\nu}\Theta \tag{4}$$

and if $Pr = \nu/\kappa$ is the Prandtl number (1) and (2) can be combined to give

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)\left(\frac{\partial^2}{\partial z^2}-a^2-\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2}-a^2-Pr\frac{\partial}{\partial t}\right) W = -a^2(R+S\cos\Omega t)\, W. \tag{5}$$

Here R is the mean Rayleigh number, so clearly, if $R+S$ is less than $\bar{R}_c$ where $\bar{R}_c$ is the least Rayleigh number for which an unstable solution exists in the corresponding problem with $S \equiv 0$, the temperature gradient is always stable and the layer of fluid can never become unstable. If $R-S > \bar{R}_c$ the layer is always unstable, while if R lies between $\bar{R}_c - S$ and $\bar{R}_c + S$ it can pass through a phase in which it is temporarily unstable followed by another in which the disturbances decay again. The problem is to find the circumstances under which a disturbance can, over a complete cycle, experience a net gain in amplitude.

The problem to be considered is the simplest possible one, which, although sometimes rather hypothetical when regarded as a laboratory experiment, has the merit of analytical simplicity so that some insight may be gained into the possible behaviour of such a layer in more realistic (but mathematically less tractable) situations. In the example studied here, the upper and lower surfaces are considered to be given specified temperatures. Equation (4) then shows that

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)\left(\frac{\partial^2}{\partial z^2}-a^2-\frac{\partial}{\partial t}\right) W = 0 \quad \text{at} \quad z = 0,\, 1;$$

the normal component of velocity at the boundaries must vanish, so that

$$W = 0 \quad \text{at} \quad z = 0,\, 1,$$

and in addition we shall suppose that the boundaries are free with the consequence that only the normal component of the stress tensor can be non-zero. This implies (Chandrasekhar 1961, p. 22) that

$$\frac{\partial^2 W}{\partial z^2} = 0.$$

All these boundary conditions can be satisfied if W is of the form

$$\sin n\pi z\, F(t);$$

equation (5) is satisfied provided that

$$Pr\frac{d^2F}{dt^2} + (Pr+1)\,N\frac{dF}{dt} + \left\{N^2 - \frac{a^2}{N}(R+S\cos\Omega t)\right\} F = 0, \tag{6}$$

where $N = a^2 + n^2\pi^2$. If we write $\sigma = -(Pr+1)\,N/2Pr$, F can be expressed in the form

$$e^{\sigma t} M(t)$$

so that M is a solution of Mathieu's equation

$$\frac{d^2M}{dt^2} - \chi(t)\,M = 0 \tag{7}$$

when

$$\chi(t) = \frac{N^2(Pr-1)^2}{4\,Pr^2} + \frac{Ra^2}{N\,Pr} + \frac{Sa^2}{N\,Pr}\cos\Omega t.$$

A result of particular importance is Floquet's theorem (see Abramowitz & Stegun 1965, p. 727), which implies that, unless n is an integer, the general solution of Mathieu's equation can be expressed in the form

$$A\,e^{imt}P(t) + B\,e^{-imt}P(-t),$$

where m may be real or complex and the function P is periodic with period $2\pi/\Omega$. The condition for neutral stability is thus

$$\frac{(Pr+1)}{2Pr}N = |\mathscr{I}m|$$

and this relation divides the (a^2, R, S) space into a region in which the corresponding solution of (6) is stable and others in which it is not. It is possible to obtain an approximation to the neutral stability surfaces by considering asymptotic solutions of Mathieu's equation given by the Horn–Jeffreys method (Jeffreys 1924). This shows that there exist solutions of (7) for large values of $\Omega^{-1}\|\chi\|^{\frac{1}{2}}$ of the asymptotic form

$$\chi(t)^{-\frac{1}{4}}\exp\left\{\pm\int^t \chi(t)^{\frac{1}{2}}\,dt\right\}.$$

It should be noticed that this solution shows that the dominant effect is the integral of instantaneous values of the coefficient of M in (7) modified only by a relatively unimportant factor $\chi^{-\frac{1}{4}}$. In general the solution has an oscillatory character for part of a period, followed by an appropriate combination of exponential growth and decay. The asymptotic solution fails when $\chi(t) = 0$, but the transition has been thoroughly investigated by Jeffreys (1924), Strutt (1943) and others. The worst possible case, which is the one of principal concern here, is the one in which a solution that is initially growing exponentially enters an oscillatory phase and when it leaves it again continues in exponential growth with its amplitude altered only by the changes in $\chi^{-\frac{1}{4}}$, which affect all solutions equally. All the other solutions grow less rapidly than this one, and so an equation for the envelope of the neutral stability surfaces (obtained previously by Rosenblat & Herbert 1970) is given by

$$\frac{Pr+1}{2Pr}N = \frac{\Omega}{2\pi}\mathscr{R}\left\{\int_0^{2\pi/\Omega}\chi(t)^{\frac{1}{2}}\,dt\right\}; \tag{8}$$

when S is small enough for χ to be positive throughout the cycle, this is the same criterion as that employed by Gresho & Sani (1970). The actual regions in the (a^2, R, S) space in which the unstable solutions exist consist of an infinite number of tongues all tangent to (8) and lying above it. Their exact location will depend on the particular values for Ω, Pr and n, and their boundaries are given by

$$\cosh\{\pi(Pr+1)\,N/Pr\,\Omega\} = \cosh\left\{\mathscr{R}\int_0^{2\pi/\Omega}\chi^{\frac{1}{2}}\,dt\right\}\cos\left\{\mathscr{R}\int_0^{2\pi/\Omega}(-\chi)^{\frac{1}{2}}\,dt\right\} \tag{9}$$

(Strutt 1932, p. 234), thus lying above (8).

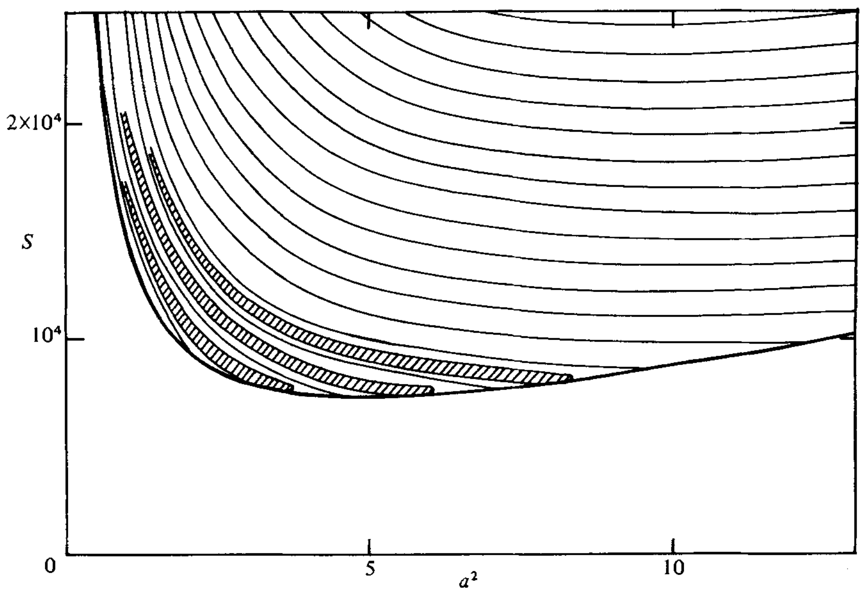

FIGURE 1. This graph shows the envelope given by (9) of the marginal stability curves when $Pr = 7$, $n = 1$ and $R = 0$ as a thick line. The isolated thin lines show the centre-lines of stable areas. Between each adjacent pair lies an unstable tongue. For clarity, only parts of a few have been drawn; the unstable regions of those which have been shown are shaded.

We need only be concerned with unstable solutions given by this approximation, and for these a least estimate of $\|\chi\|^{\frac{1}{2}}$ is provided by (8). Larger values of R correspond to more unstable solutions with larger values of $\|\chi\|^{\frac{1}{2}}$. The neutral stability surface is thus given approximately by (8) provided that

$$\frac{Pr+1}{2Pr}\frac{N}{\Omega} \geqslant \frac{\pi^2(Pr+1)}{2Pr\Omega} \gg 1, \tag{10}$$

even though the stable solutions at small values of R are not given accurately by the asymptotic solution. Note that condition (10) is a consequence of (3), since the latter can be written $\Omega Pr \ll 1$.

Two examples will be considered to illustrate the general characteristics of the solutions of (8) and (9). The first is one in which there is no mean temperature gradient so that $R = 0$ but in which the value of S is considered to be adjustable, while in the second example S is fixed but R is variable. In each case we are concerned only with plane sections of the general stability surfaces.

The first is illustrated in figure 1, which shows the envelope and tongues using approximation (9) when $Pr = 7$, $n = 1$ and $R = 0$ so that the mean temperature gradient is zero; Ω has been taken to be equal to one in order to show the tongues, which are very closely spaced indeed when $\Omega \ll 1$. The asymptotic approximation is nonetheless reasonably satisfactory. The equation of the envelope (which is unaffected by the particular choice of small value for Ω) is approximately

$$7Sa^2 = 8{\cdot}6 \times 9(a^2+\pi^2)^3$$

with a minimum of S, $S_c \simeq 7300$, at $a_c^2 = \frac{1}{2}\pi^2$. Similar curves may be drawn for different values of n for which it is found that the minimum values of S attained by the envelopes are approximately $7300n^4$, so the least stable mode is the first. The error δS introduced by using the minimum values of S on the envelope instead of the minimum value on the boundaries of the tongues is of the order of 1030Ω, that is, an error of about $14\Omega\%$. Since this estimate is based on the least critical values of S at $a_c^2 = \frac{1}{2}\pi^2$ and the envelope is relatively flat there, the actual error for S_c (though not for a_c^2) will usually be considerably less.

The general characteristics of the solution are similar for other problems of the type in which the layer passes alternately through phases in which the layer is intrinsically stable, then unstable, and for part of the time passes through an oscillatory phase. The most useful observation from the point of view of the analytical investigation of more sophisticated problems is that the stability of the layer can be determined with considerable accuracy by considering only the minimum of the envelope of the bounding tongues in the (a^2, S) plane in which the disturbances are unstable.

The second example is one in which S is considered to be given but R is variable. It is possible to recover the results of Venezian (1969), Rosenblat & Herbert (1970), and Gresho & Sani (1970) by considering the case when $S \ll R$ so that the problem is not very different from the one in which the temperature gradient is steady. If we take $n = 1$ and $\Omega \ll 1$ the neutral stability curve is given by

$$R = \frac{N^3}{a^2} + \frac{a^2 Pr}{2(Pr+1)^2 N^3} S^2,$$

which again has a minimum at $a^2 = \frac{1}{2}\pi^2$ and a corresponding critical value for R of

$$\tfrac{27}{4}\pi^4 + \frac{2\,Pr}{27(Pr+1)^2}\frac{S^2}{\pi^4}$$

(obtained by both Venezian and Gresho & Sani) so that in fact the presence of sufficiently small fluctuations in the temperature gradient has a stabilizing effect. In this instance the curve is not just an envelope but in fact is everywhere a division of the plane between stable and unstable solutions.

3. The neutral stability condition

In order to determine the combinations of R and S for which the layer is stable to disturbances of any wavenumber, write the bounding surface (8) as either

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (r - s\sin t)^{\frac{1}{2}}\, dt = \pi, \quad s \leqslant r, \tag{11}$$

or

$$\int_{-\frac{1}{2}\pi}^{\arcsin r/s} (r - s\sin t)^{\frac{1}{2}}\, dt = \pi, \quad s > r, \tag{12}$$

where

$$r = \left(\frac{Pr-1}{Pr+1}\right)^2 + \frac{4\,Pr\,a^2 R}{(Pr+1)^2 N^3}, \quad s = \frac{4\,Pr\,a^2 S}{(Pr+1)^2 N^3}.$$

These equations describe a neutral stability envelope in the (a^2, R, S) space, which is illustrated in figure 2. The stable values of R and S are those for which any line

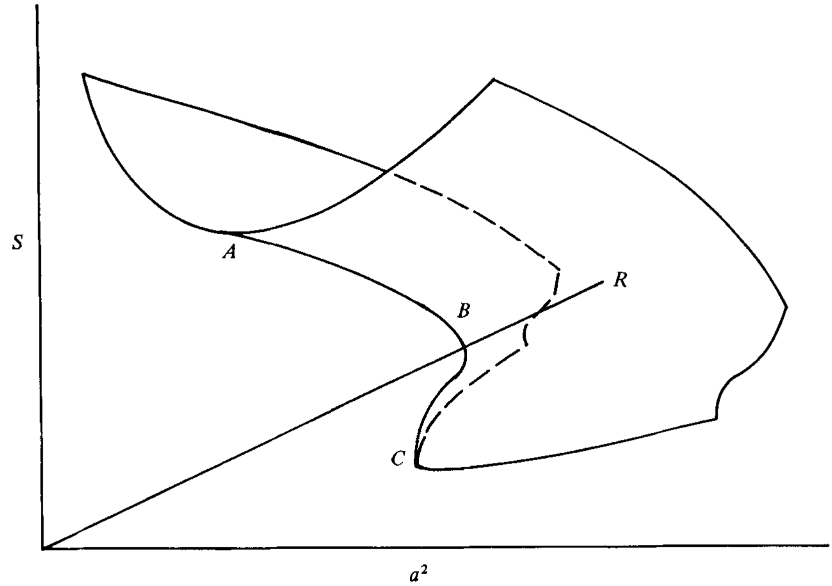

FIGURE 2. This illustrates the relation between the stability surface in the (a^2, R, S) space and the curve relating critical values of r and s shown in figure 3. The curve is here represented by the line ABC on the neutral stability surface, and consists of those points lying nearest to $R = 0$ (when $R > 0$). They all lie in the plane $a^2 = \frac{1}{2}\pi^2$.

parallel to the a^2 axis fails to cut the envelope. This may be true for some others as well, though in general the error is not likely to be large except perhaps for values of r and s near to the region where transition from formula (11) to formula (12) takes place.

It is clear from figure 2 that the line ABC on this surface (where lines parallel to the a^2 axis are all tangent to the surface) relates critical values of R and S, and that at these points $\partial R/\partial a^2$ and $\partial S/\partial a^2$ both vanish. Under these conditions $\partial r/\partial a^2$ and $\partial s/\partial a^2$ are both proportional to $\pi^2 - 2a^2$ and so for both (11) and (12) the critical wavenumber always occurs at $a_c^2 = \frac{1}{2}\pi^2$, irrespective of which equation applies, a result found to be approximately true by Rosenblat & Tanaka (1971) up to moderate values of s/r for the problem with rigid boundaries. The corresponding values of r_c and s_c are respectively

$$\left(\frac{Pr-1}{Pr+1}\right)^2 + \frac{16 Pr\, R}{27(Pr+1)^2\pi^4} \quad \text{and} \quad \frac{16\, Pr\, S}{27(Pr+1)^2\pi^4}. \tag{13}$$

When $S = 0$ so that $s = 0$ the case of a steady mean temperature gradient is obtained with the critical value for R of $27\pi^4/4$. As S increases relation (11) initially applies and since the left-hand side is a decreasing function of s it follows that the critical Rayleigh number for instability must rise. This continues to be true at least until $r_c = s_c = \pi^2/8$, which corresponds to

$$R = \frac{27\pi^4}{16\, Pr}\left\{\frac{\pi^2}{8}(Pr+1)^2 - (Pr-1)^2\right\}, \quad S = \frac{27\pi^6(Pr+1)^2}{128\, Pr}.$$

For greater values of S the critical value of R has to be determined from (12).

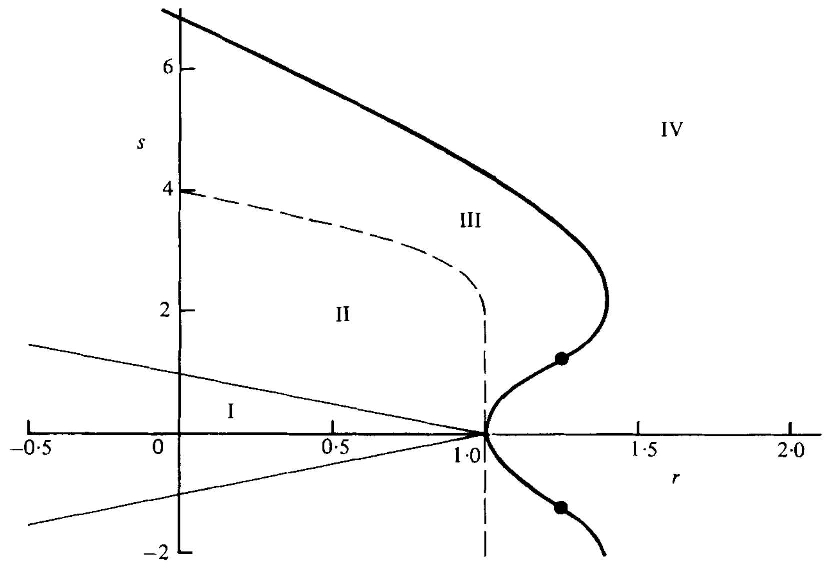

FIGURE 3. The relation between r and s in the plane $a^2 = \frac{1}{2}\pi^2$ of the (a^2, R, S) space is shown by the thick curve. It is a relation between the critical values of R and S separating solutions which are stable for all values of a^2 from those which are (apart from some reservations mentioned in the text) unstable for some values of a^2. For values of R and S giving r and s to the right of the curve the layer is unstable while for values to the left it is stable according to the linear analysis; dots show the points at which transition from (11) to (12) occurs. Homsy's criterion for global stability is shown by the thin line and his criterion for asymptotic stability for $Pr = 1$ when $r > 0$ by the broken line.

It is at this stage that the solutions can enter an oscillatory phase with the consequence that not all points beyond the envelope (8) correspond to unstable solutions. Initially the left-hand side of (12) is increasing and R_c continues to rise. However, in due course a maximum appears and R_c begins to fall as S is increased. It then continues to decrease without bound, even passing into negative values. This means that if the amplitude of the variations in the temperature gradient are very big indeed a progressively larger intrinsically stable mean gradient is required to ensure stability over a period. The effect of large-amplitude fluctuations (in so far as these can be meaningfully achieved in practice) is to destabilize the layer. The values of R and S for which the maximum critical value of R occurs can be obtained from r_c and s_c above when they satisfy (12), and also

$$\int_{-\frac{1}{2}\pi}^{\arcsin r/s} \frac{\sin t}{(r - s\sin t)^{\frac{1}{2}}}\, dt = 0.$$

Approximate values are $r = 1{\cdot}4$ and $s = 2{\cdot}1$.

Equation (11) can be rewritten as

$$E\left(\frac{2s}{r+s}\right) = \frac{\pi}{2(r+s)^{\frac{1}{2}}}$$

and (12) as

$$\pi(\tfrac{1}{2}s)^{\frac{1}{2}} = (r-s)\,K\left(\frac{r+s}{2s}\right) + 2sE\left(\frac{r+s}{2s}\right) \quad (r+s \geqslant 0),$$

where K and E are complete elliptic integrals of the first and second kind respectively (Abramowitz & Stegun, p. 590). The first form was found by Rosenblat & Herbert, but they did not carry their investigations further. From these relations it is possible to calculate the dependence of the critical values of r and s on each other. The resulting curve (which, as was pointed out above, lies in the plane $a^2 = \frac{1}{2}\pi^2$) is shown in figure 3; the corresponding values for R and S for given Prandtl number can be recovered from (13). It is illuminating to compare it with the exact neutral stability curves computed by Yih & Li, and displayed in figures 2 and 3 of their paper. Although the problem considered by them was rather different it is quite clear that there is a strong qualitative similarity with the present result and it is reasonable to suppose that the main cause in their case as well as here is the variation of the temperature gradient in time, rather than (as might be possible in their problem but not in ours) its consequent variation in space.

4. Discussion

It is of considerable interest to compare the marginal stability curve found here for the problem with two free boundaries, with the two general criteria found by Homsy (1974) and discussed in some detail for this particular example. His criterion for *global stability* is equation (4.6) of his paper and has the simple interpretation that the layer must be stable at all times by the standard test appropriate to the steady problem. Expressed in terms of the values of r and s when $a^2 = \frac{1}{2}\pi^2$ by means of (13), it becomes

$$r \leqslant 1-|s|.$$

His analysis presupposes that the Rayleigh number is positive, but the result almost certainly generalizes; the pair of lines corresponding to it is shown by the continuous thin line in figure 2.

Homsy's condition for asymptotic stability gives rise to a much more complicated relation between the critical values of r and s, and when rewritten in the notation used here becomes

$$2\pi = \min_{\lambda \geq 0} \int_0^{2\pi} \left\{ \xi + (r-\xi) \left[\frac{1+\lambda+s\sin t/(r-\xi)}{2\lambda^{\frac{1}{2}}} \right]^2 \right\}^{\frac{1}{2}} dt, \tag{14}$$

where

$$\xi = \left(\frac{1-Pr}{1+Pr} \right)^2.$$

He gives forms for the relation in the special cases $Pr \to 0$, $Pr \to \infty$ and $Pr = 1$, but in fact the derivations given for these relations are in all cases valid for limited ranges of r and s. The one where $Pr = 1$ for example gives $r = 1$ for $|s| \leqslant 1$; Homsy's analysis presumes that R is necessarily positive so that the condition is not automatically valid for large values of s. In general, the relation between r and s will depend on Pr and is not readily obtainable. It is possible to proceed further with the special case for which $Pr = 1$ however, since in that instance $\xi = 0$ and for $|s| > r$ the relation becomes

$$\frac{1}{|s|^{\frac{1}{2}}} = \left(\frac{r}{|s|} \right)^{\frac{1}{2}} \quad \text{if} \quad 2r \geqslant |s|$$

or

$$\frac{1}{|s|^{\frac{1}{2}}} = \min\left\{\frac{1}{2(1-r/|s|)^{\frac{1}{2}}}, \min_{1 \geqslant \mu \geqslant r/|s|} \frac{\mu \sin\mu + (1-\mu^2)^{\frac{1}{2}}}{\pi(\mu - r/|s|)^{\frac{1}{2}}}\right\} \quad \text{if} \quad 2r \leqslant |s|. \tag{15}$$

This relation can be investigated numerically when $r > 0$ (the only circumstance for which the derivation of (14) is valid) and the resulting curve is shown by the broken line in figure 3. Although it is not possible to show the difference on the scale of the graph, the first term in (15) is smaller at the left-hand end of the curve, while the second term is less by a small margin at the right-hand end. As a result the curved portion is a little closer to the thick curve than would be so if only the parabolic relation implied by the first term of (15) were employed.

The three curves shown in figure 3 divide the (r, s) plane into four regions labelled I–IV, and it is possible to provide an interpretation of them. For all values of r and s in I the layer is globally stable whereas in II this is not necessarily so, although any disturbance of whatever amplitude is guaranteed ultimately to decay to zero even though it may at certain times have increasing energy. In region IV disturbances are unstable by all criteria; in region III, however, there is no assurance that they will decay to zero, even though they will do so if their amplitude is sufficiently small for the linearized theory to be valid. In practice it seems likely that what will be observed will be intermittent growth of individual disturbances followed by decay to a level which may not be detectable. These could, however, lead to substantial and unpredictable effects on the transport of heat, mass and momentum in the system (Davis, p. 58).

In conclusion, we have seen that the linearized theory shows that modulation at large amplitude has a strongly destabilizing effect, and in consequence there is a maximum degree of enhancement of stability; this effect can be caused by the periodicity of the temperature gradient in time alone.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1965 *Handbook of Mathematical Functions*. Dover.

CHANDRA, K. 1938 Instability of fluids heated from below. *Proc. Roy. Soc.* A **164**, 231–242.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Clarendon.

CURRIE, I. G. 1967 The effect of heating rate on the stability of stationary fluids. *J. Fluid Mech.* **29**, 337–347.

DAVIS, S. H. 1976 The stability of time-periodic flows. *Ann. Rev. Fluid Mech.* **8**, 57–74.

GOLDSTEIN, A. W. 1959 Stability of a horizontal fluid layer with unsteady heating from below and time-dependent body forces. *N.A.S.A. Tech. Rep.* no. R-4.

GRAHAM, A. 1933 Shear patterns in an unstable layer of fluid. *Phil. Trans. Roy. Soc.* A **232**, 285–296.

GRESHO, P. M. & SANI, R. L. 1970 The effect of gravity modulation on the stability of a heated fluid layer. *J. Fluid Mech.* **40**, 783–806.

HOMSY, G. M. 1974 Global stability of time-dependent flows. Part 2. Modulated fluid layers. *J. Fluid Mech.* **62**, 387–404.

JEFFREYS, H. 1924 On certain approximate solutions of linear differential equations of the second order. *Proc. Lond. Math. Soc.* **23**, 428–436.

ROSENBLAT, S. & HERBERT, D. M. 1970 Low-frequency modulation of thermal instability. *J. Fluid Mech.* **43**, 385–398.

ROSENBLAT, S. & TANAKA, G. A. 1971 Modulation of thermal convective instability. *Phys. Fluids* **14**, 1319–1322.

STRUTT, M. J. O. 1932 Lamésche-, Mathieusche- und verwandte Functionen in Physik und Technik. *Ergebn. Math.* **1**, 199–323.

STRUTT, M. J. O. 1943 Eigenwaarde-krommen bij problemen van Hill. II. Asymptotisch verloop der krommen. *Versl. gewone Vergad. Afd. Natuurk. K. ned. Akad. Wet.* **52**, 212–222.

SUTTON, O. G. 1950 On the stability of a fluid heated from below. *Proc. Roy. Soc.* A **204**, 297–309.

VENEZIAN, G. 1969 Effect of modulation on the onset of thermal convection. *J. Fluid Mech.* **35**, 243–254.

YIH, C.-S. & LI, C.-H. 1972 Instability of unsteady flows or configurations. Part 2. Convective instability. *J. Fluid Mech.* **54**, 143–152.